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DIFFUSION IN COMPOSITE LAYERS WITH AUTOMATIC SOLUTION OF THE EIGENVALUE PROBLEM

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Abstract—The analytical treatment of transient heat conduction problems for one-dimensional multilayered composites by the orthogonal expansion technique requires the solution of a corresponding eigenvalue problem if this analytical solution is to be implemented for practical purposes. Such an eigenvalue problem is not of the conventional Sturm–Liouville type because of the discontinuities of the coefficient functions. Its solution with conventional techniques is not guaranteed from missing eigenvalues in the course of the computation.

An analytical solution of one transient heat conduction problem in one-dimensional multilayered slabs, cylinders and spheres is presented, which implements a safe algorithm for the automatic computation of the eigenvalues and the eigenfunctions of the resulting Sturm–Liouville type system.

NOMENCLATURE

A, accuracy of the eigenvalues, Step 1 of the algorithm;
C_k, D_k, constants, equation (4a);
I, number of eigenvalues required, Step 1 of the algorithm;
I_0, maximum number of iterations, Step 1 of the algorithm;
J_0, Y_0, Bessel functions of the first and second kind, respectively;
$[K(\mu)]$, matrix, equations (9a) and (9c);
$[K^{\Delta}(\mu)]$, the triangulated form of $[K(\mu)]$;
L, order of the power series approximation, Section 1;
$N(\mu)$, total number of eigenvalues, equation (10);
$N_0(\mu)$, total number of eigenvalues of the degenerate system, equation (12b);
$N_{0k}(\mu)$, total number of eigenvalues of a single layer, equation (12c);
N_i, normalization integral for the ith eigenfunction, equation (12b);
$P_k(\mu, x), Q_k(\mu, x)$, functions, equations (5d) and (5e), Table 3;
$T_k(x, t)$, temperature distribution in the kth layer;
$U_k(\mu, x), V_k(\mu, x)$, functions, equations (4c)–(4f), Table 2;
$W_m(\lambda x)$, function, equation (16);
a_0, a_k, b_k, a_n, coefficients, equations (9d)–(9g);
c_k, heat capacity of the kth layer;
d_k, the main diagonal of the matrix $[K^{\Delta}(\mu)]$, equation (13);
$f_k(x), f_i$, functions defined in equations (1f) and (2c), respectively;
h_k, film coefficient at the interface $x = x_k$;
i_0, iteration counter, Step 3 of the algorithm;
k_k, thermal conductivity of the kth layer;
m, geometry index, explained in Section 2;
n, total number of layers;
r, increment of the eigenvalue parameter, Step 2 of the algorithm;
$s([K(\mu)])$, the 'sign count' of the matrix $[K(\mu)]$, [1, 2];
t, time;
$u_k(\mu, x), v_k(\mu, x)$, functions, Table 1;
x, space coordinate;
z, parameter, Section 5.

Greek symbols

$\alpha_0, \beta_0, \alpha_n, \beta_n$, parameters, equations (1b) and (1e);
α^*, thermal diffusivity;
λ, parameter, equation (16);
μ, eigenvalue parameter;
μ_i, the ith eigenvalue;
π, $= 3.1415926$;
ρ_k, specific gravity;
$\Psi_{ik}(\mu_i, x)$, the ith eigenfunction in the kth layer;
Ψ_k^*, value of the corresponding eigenfunction at the interface $x = x_k$;

Φ_0, Φ_n, defined in equations (1b) and (1e);
ω_k, defined in Section 3;
$\Omega_i(x_0), \Omega_i(x_n)$, defined by equations (2d) and (2e).

Subscripts

i, the order of the eigenvalue μ_i and the corresponding eigenfunction $\Psi_i(\mu_i, x)$, $i = 1, 2, 3, \ldots$;
k, dummy variable denoting any of the layers, $k = 1, 2, \ldots, n$.

1. INTRODUCTION

THE TRANSIENT temperature distribution in a composite medium consisting of several layers of different physical properties in contact has numerous applications in science and engineering. Various methods are available for the analysis of such problems: the orthogonal expansion technique and the Green's function approach [3–14], the adjoint solution technique [8, 15], the Laplace transform technique [16–19], and finite integral transforms [20–29].

It is obvious that there are various ways to derive a formal solution of the problem considered. But to implement this solution for practical purposes one has to realize it numerically and this leads one to the necessity of solving the corresponding Sturm–Liouville eigenvalue–eigenfunction problem, which is not of the conventional type because of the discontinuity of the coefficient functions.

The safe and fast computation of the eigenvalues and the corresponding eigenfunctions is a tricky piece of work and the discussion of this problem has often been avoided, as can be seen from most of the references mentioned. As a matter of fact, Mulholland and Cobble [10] developed an algorithm and presented a detailed numerical example for a multilayered slab, while Lockwood and Mulholland [24] did the same for a multilayered cylinder. The method developed and implemented by these authors allows for the computation of the whole numerable spectrum of the problem, but one is not guaranteed from missing eigenvalues in the course of computation. Horgan *et al.* [30] state that "considerable emphasis has been placed on the development of computational schemes for estimating eigenvalues and eigenfunctions for such problems. These efforts have met with serious difficulties due to the nonsmoothness of the coefficients and the resulting spectrum irregularities," and later "a complete spectral theory for Sturm–Liouville problems with discontinuous coefficients has not yet been established." Utilizing integral equation methods, these authors find lower bounds for the eigenvalues of discontinuous coefficient Sturm–Liouville problems. Almost ten years earlier, Ramkrishna and Amundson [14], in a very interesting paper, have shown that if the coefficients of a Sturm–Liouville problem are not smooth in a finite number of points in the interior of a finite interval, the corresponding Sturm–Liouville operator is symmetric, which is also valid for the Green's function representing its inverse. As far as the integral of the Green's function is a self-adjoint operator in the corresponding Hilbert space, it follows that the Sturm–Liouville operator has a numerable set of eigenvalues and a complete orthonormal family of eigenfunctions forming a basis there. Stating that the purpose of their work is to "expound a convenient formalism for the solution of the boundary value problem under discussion", these authors "avoid the discussion of computational aspects such as the evaluation of the eigenvalues" from the corresponding eigencondition, although several interesting examples from the field of heat and mass transfer have been considered. Hodges [31] developed a procedure based on the method of Ritz to compute the upper bounds of the eigenvalues of a discontinuous coefficients' Sturm–Liouville problem, and has presented numerical examples for their computation and the evaluation of the corresponding eigenfunctions. His method allows one to estimate only a finite number of eigenvalues of the numerable spectrum of the problem, and with the increase of the order L of the power series approximation the numerical stability of the method decreases thus "requiring double precision arithmetics to solve the eigenvalue problem on a CDC 7600 computer for L greater than 6." Horgan and Nemat-Nasser [32, 33] also make use of variational methods to estimate bounds for the eigenvalues of problems of the type under consideration.

Wittrick and Williams [1, 2] (the second paper reviewing all their previous results) developed an extremely efficient (in terms of computer resources) algorithm for the safe and automatic computation of the natural frequencies and buckling loads of linear elastic skeletal structures. It permits one to estimate exactly how many natural frequencies lie below any fixed value of the frequency parameter without calculating them explicitly, and thus to converge on any required eigenfrequency to any reasonably-chosen (in the sense of the computer word-length) accuracy.

Recently, the algorithm of Wittrick and Williams [1, 2] was adapted by Mikhailov and Vulchanov [34] for the solution of linear Sturm–Liouville problems. The computational procedure developed in ref. [34] can be applied directly for the analysis of multilayered slabs.

The purpose of the present paper is to generalize the ideas from ref. [34], applying them for the cases of cylindrical and spherical geometries. Next, utilizing the orthogonal series solution for 1-dim. multilayered composites derived in the forthcoming book [35], the transient temperature distribution in such bodies is computed to illustrate the effectiveness of the method discussed.

2. FORMULATION OF THE TRANSIENT HEAT CONDUCTION IN ONE-DIMENSIONAL MULTILAYERED COMPOSITES

Consider a composite medium consisting of n parallel layers of slabs, cylinders or spheres, as shown in

Fig. 1. For generality, it is assumed that contact resistances at the interfaces [10, 36, 37] are present together with convection from the outer boundaries. Let h_k be the film coefficient at the interface $x = x_k$, $k = 1,2,3,\ldots,(n-1)$. Each layer is homogeneous, isotropic and has thermal properties (i.e. ρ_k, c_k and k_k) that are constant within each layer and different from those of the adjacent layers. Initially, each layer is at temperature $T(x,0) = f_k(x)$ in $x_{k-1} < x < x_k$, $k = 1, 2,\ldots,n$. For times $t > 0$, heat is transferred from the two outer boundaries according to boundary conditions of the first, second or third kind. There is no heat generation in the medium. The mathematical formulation of this heat conduction problem governing the temperature distribution $T_k(x,t)$, $k = 1,2,\ldots,n$, for times $t > 0$ is given as

$$x^{1-2m}\frac{\partial T_k(x,t)}{\partial t} = \alpha_k^* \frac{\partial}{\partial x}\left(x^{1-2m}\frac{\partial T_k(x,t)}{\partial x}\right) \quad (1a)$$

in $x_{k-1} < x < x_k$ for $t > 0$ and $k = 1,2,\ldots,n$, where $\alpha_k^* = k_k/(\rho_k c_k)$ and $m = 1/2$ for slab, $m = 0$ for cylinder, $m = -1/2$ for sphere; subject to the boundary conditions

$$\alpha_0 T_1(x_0,t) - \beta_0 k_1 x_0^{1-2m}\frac{\partial T_1(x_0,t)}{\partial x} = \Phi_0 \quad (1b)$$

at the outer boundary $x = x_0$ for $t > 0$,

$$-k_k\frac{\partial T_k(x_k,t)}{\partial x} = h_k[T_k(x_k,t) - T_{k+1}(x_k,t)] \quad (1c)$$

$$k_k\frac{\partial T_k(x_k,t)}{\partial x} = k_{k+1}\frac{\partial T_{k+1}(x_k,t)}{\partial x} \quad (1d)$$

at the interfaces $x = x_k$, $k = 1,2,\ldots,(n-1)$ for $t > 0$,

$$\alpha_n T_n(x_n,t) + \beta_n k_n x_n^{1-2m}\frac{\partial T_n(x_n,t)}{\partial x} = \Phi_n \quad (1e)$$

at the outer boundary $x = x_n$ for $t > 0$. The initial conditions are given by

$$T_k(x,0) = f_k(x); \quad t = 0, \qquad x_{k-1} < x < x_k, \quad k = 1,2,\ldots,n. \quad (1f)$$

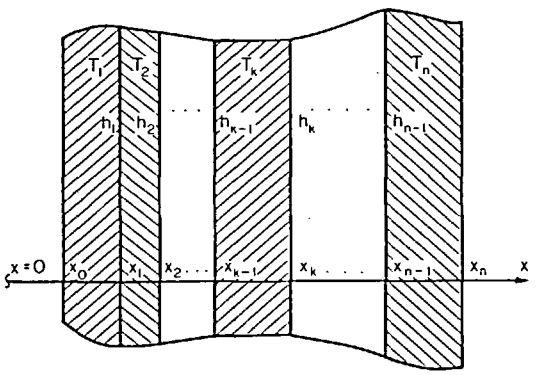

FIG. 1. An n-layered composite: slab ($m = 1/2$), cylinder ($m = 0$) and sphere ($m = -1/2$).

By appropriate choice of the values of the parameters α_0, β_0, α_n and β_n, various combinations of boundary conditions of the first, second or third kind are obtainable at the two outer boundaries. The special case $\alpha_0 = \alpha_n = 0$ and $\beta_0 = B_n = 1$ at the two outer boundaries is not considered here.

The physical significance of the interface boundary conditions [equations (1c) and (1d)] is as follows: the finite value of the film coefficient h_k, $k = 1,2,\ldots,(n-1)$ in the boundary condition [equation (1c)], implies a discontinuity of the temperature at the corresponding interface. The boundary condition [equation (1d)] states that the heat flux is continuous at the same interface. For the special case $h_k \to \infty$, equation (1c) reduces to

$$T_k(x_k,t) = T_{k+1}(x_k,t); \quad t > 0, \qquad x = x_k, \quad k = 1,2,\ldots,(n-1) \quad (1g)$$

which means continuity of the temperature across the interface at $x = x_k$, or perfect thermal contact there.

3. THE FORMAL SOLUTION FOR A n-LAYERED COMPOSITE

The formal solution of the problem, defined by equations (1) has been derived and discussed in detail in a forthcoming book [35, Ch. 8]. The final result obtained there has the form

$$\begin{aligned}T_k(x,t) = {} & \left\{\left[\beta_n + \alpha_n\left(\frac{1}{k_k}\int_x^{x_k} x^{2m-1}\,dx + \sum_{i=k+1}^{n}\frac{1}{k_i}\int_{x_{i-1}}^{x_i} x^{2m-1}\,dx + \sum_{i=k}^{n-1}\frac{x_i^{2m-1}}{h_i}\right)\right]\Phi_0\right. \\ & + \left[\beta_0 + \alpha_0\left(\frac{1}{k_k}\int_{x_{k-1}}^{x} x^{2m-1}\,dx + \sum_{i=1}^{k-1}\frac{1}{k_i}\int_{x_{i-1}}^{x_i} x^{2m-1}\,dx + \sum_{i=1}^{k-1}\frac{x_i^{2m-1}}{h_i}\right)\right]\Phi_n\Bigg\} \\ & \times\left[\alpha_0\beta_n + \alpha_n\beta_0 + \alpha_0\alpha_n\left(\sum_{k=1}^{n}\frac{1}{k_k}\int_{x_{k-1}}^{x_k} x^{2m-1}\,dx + \sum_{k=1}^{n-1}\frac{x_k^{2m-1}}{h_k}\right)\right]^{-1} \\ & + \sum_{i=1}^{\infty}\frac{\Psi_{ik}(\mu_i,x)}{N_i}\exp(-\mu_i^2 t) \\ & \times\left\{f_i - \frac{1}{\mu_i^2}[\Phi_0\Omega_i(x_0) + \Phi_n\Omega_i(x_n)]\right\}; \\ & k = 1,2,\ldots,n \end{aligned} \quad (2a)$$

where the functions N_i, f_i, $\Omega_i(x_0)$ and $\Omega_i(x_n)$ are defined by

$$N_i = \sum_{k=1}^{n}\rho_k c_k\int_{x_{k-1}}^{x_k} x^{1-2m}\Psi_{ik}^2(\mu_i,x)\,dx, \quad (2b)$$

$$f_i = \sum_{k=1}^{n}\rho_k c_k\int_{x_{k-1}}^{x_k} x^{1-2m}\Psi_{ik}(\mu_i,x)f_k(x)\,dx, \quad (2c)$$

Table 1. The linearly independent solutions $u_k(\mu, x)$ and $v_k(\mu, x)$ of equation (3a) for slab, cylinder and sphere

Geometry	m	$u_k(\mu, x)$	$v_k(\mu, x)$
Slab	1/2	$\cos(\omega_k x)$	$\sin(\omega_k x)$
Cylinder†	0	$J_0(\omega_k x)$	$Y_0(\omega_k x)$
Sphere†	$-1/2$	$\sin(\omega_k x)/(\omega_k x)$	$\cos(\omega_k x)/(\omega_k x)$

† $x_0 > 0$; here and in what follows, the derivations are valid for hollow multilayered cylinders and spheres.

$$\Omega_i(x_0) = \frac{\Psi_{i1}(\mu_i, x_0) + k_1 x_0^{1-2m}\Psi'_{i1}(\mu_i, x_0)}{\alpha_0 + \beta_0}, \quad (2d)$$

$$\Omega_i(x_n) = \frac{\Psi_{in}(\mu_i, x_n) - k_n x_n^{1-2m}\Psi'_{in}(\mu_i, x_n)}{\alpha_n + \beta_n}. \quad (2e)$$

In equations (2), μ_i and $\Psi_i(\mu_i, x)$, $i = 1, 2, \ldots$ are the eigenvalues and the eigenfunctions of the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{1-2m}\frac{d\Psi_k(\mu, x)}{\mathrm{d}x}\right] + \omega_k^2 x^{1-2m}\Psi_k(\mu, x) = 0 \quad (3a)$$

in $x_{k-1} < x < x_k$, $k = 1, 2, \ldots, n$ and $\omega_k = \mu(\alpha_k^*)^{-1/2}$; subject to the boundary conditions

$$\alpha_0\Psi_1(\mu, x_0) - \beta_0 k_1 x_0^{1-2m}\Psi'_1(\mu, x_0) = 0, \quad (3b)$$

$$-k_k x^{1-2m}\Psi'_k(\mu, x_k) = h_k[\Psi_k(\mu, x_k) - \Psi_{k+1}(\mu, x_k)];$$

$$k = 1, 2, \ldots, (n-1), \quad (3c)$$

$$k_k\Psi'_k(\mu, x_k) = k_{k+1}\Psi'_{k+1}(\mu, x_k);$$

$$k = 1, 2, \ldots, (n-1), \quad (3d)$$

$$\alpha_n\Psi_n(\mu, x_n) + \beta_n k_n x_n^{1-2m}\Psi'_n(\mu, x_n) = 0 \quad (3e)$$

where the prime denotes differentiation with respect to x. For the special case $h_k \to \infty$, the boundary condition, equation (3c), reduces to

$$\Psi_k(\mu, x_k) = \Psi_{k+1}(\mu, x_k); \quad k = 1, 2, \ldots, (n-1) \quad (3f)$$

The temperature distribution $T_i(x, t)$ in any layer k, $k = 1, 2, \ldots, n$, of the n-layered slab ($m = 1/2$), cylinder ($m = 0$) or sphere ($m = -1/2$) can be determined utilizing equations (2) if the eigenvalues μ_i and the eigenfunctions $\Psi_{ik}(\mu_i, x)$ are known. Therefore, in the next section the computational scheme for the estimation of the eigenvalues and the eigenfunctions will be discussed.

4. ANALYSIS OF THE EIGENVALUE PROBLEM

Let $u_k(\mu, x)$ and $v_k(\mu, x)$ be two linearly independent solutions of equation (3a). Table 1 shows these functions for the cases of slab, cylindrical and spherical geometries.

The eigenfunctions $\Psi_k(\mu, x)$ are generally constructed as a linear combination of the elementary solutions $u_k(\mu, x)$ and $v_k(\mu, x)$ in the form

$$\Psi_k(\mu, x) = C_k u_k(\mu, x) + D_k v_k(\mu, x). \quad (4a)$$

However, in the computational procedure to be described in what follows, the constants C_k and D_k will be replaced by the values of the eigenfunctions $\Psi_k(\mu, x_{k-1})$ and $\Psi_k(\mu, x_k)$ at the end points $x = x_{k-1}$ and $x = x_k$, $k = 1, 2, \ldots, n$, respectively. For these quantities the following notation will be used:

$$\Psi^*_{k-1} = \Psi_k(\mu, x_{k-1}); \quad \Psi^*_k = \Psi_k(\mu, x_k);$$

$$k = 1, 2, \ldots, n.$$

To do this, one evaluates equation (4a) for $x = x_{k-1}$ and $x = x_k$ and then solves the resulting linear system of two algebraic equations for C_k and D_k. When the result is introduced in equation (4a), one has

$$\Psi_k(\mu, x) = \Psi^*_{k-1}U_k(\mu, x) + \Psi^*_k V_k(\mu, x) \quad (4b)$$

where

$$U_k(\mu, x) = \frac{u_k(\mu, x)v_k(\mu, x_k) - u_k(\mu, x_k)v_k(\mu, x)}{u_k(\mu, x_{k-1})v_k(\mu, x_k) - u_k(\mu, x_k)v_k(\mu, x_{k-1})}, \quad (4c)$$

$$V_k(\mu, x) = \frac{u_k(\mu, x_{k-1})v_k(\mu, x) - u_k(\mu, x)v_k(\mu, x_{k-1})}{u_k(\mu, x_{k-1})v_k(\mu, x_k) - u_k(\mu, x_k)v_k(\mu, x_{k-1})}. \quad (4d)$$

Table 2 shows the functions $U_k(\mu, x)$ and $V_k(\mu, x)$ for the cases of slab, cylindrical and spherical geometries.

Note that the special choice of the functions $U_k(\mu, x)$

Table 2. Solutions $U_k(\mu, x)$ and $V_k(\mu, x)$, defined by equations (4c) and (4d), for slab, cylinder and sphere

Geometry	m	$U_k(\mu, x)$	$V_k(\mu, x)$
Slab	1/2	$\dfrac{\sin[\omega_k(x_k - x)]}{\sin[\omega_k(x_k - x_{k-1})]}$	$\dfrac{\sin[\omega_k(x - x_{k-1})]}{\sin[\omega_k(x_k - x_{k-1})]}$
Cylinder†	0	$\dfrac{J_0(\omega_k x)Y_0(\omega_k x_k) - J_0(\omega_k x_k)Y_0(\omega_k x)}{J_0(\omega_k x_{k-1})Y_0(\omega_k x_k) - J_0(\omega_k x)Y_0(\omega_k x_{k-1})}$	$\dfrac{J_0(\omega_k x_{k-1})Y_0(\omega_k x) - J_0(\omega_k x)Y_0(\omega_k x_{k-1})}{J_0(\omega_k x_{k-1})Y_0(\omega_k x_k) - J_0(\omega_k x_k)Y_0(\omega_k x_{k-1})}$
Sphere†	$-1/2$	$\dfrac{x_{k-1}}{x}\dfrac{\sin[\omega_k(x_k - x)]}{\sin[\omega_k(x_k - x_{k-1})]}$	$\dfrac{x_k}{x}\dfrac{\sin[\omega_k(x - x_{k-1})]}{\sin[\omega_k(x_k - x_{k-1})]}$

† $x_0 > 0$; see footnote to Table 1.

Table 3. The functions $P_k(\mu, x)$ and $Q_k(\mu, x)$, defined by equations (5d) and (5e), for slab, cylinder and sphere

Geometry	m	$P_k(\mu, x)$	$Q_k(\mu, x)$
Slab	1/2	$-\omega_k k_k \dfrac{\cos[\omega_k(x_k - x)]}{\sin(\omega_k(x_k - x_{k-1})]}$	$\omega_k k_k \dfrac{\cos[\omega_k(x - x_{k-1})]}{\sin[\omega_k(x_k - x_{k-1})]}$
Cylinder†	0	$-\omega_k k_k \dfrac{J_1(\omega_k x)Y_0(\omega_k x_k) - J_0(\omega_k x_k)Y_1(\omega_k x)}{J_0(\omega_k x_{k-1})Y_0(\omega_k x_k) - J_0(\omega_k x_k)Y_0(\omega_k x_{k-1})}$	$\omega_k k_k \dfrac{J_0(\omega_k x_{k-1})Y_1(\omega_k x) - J_1(\omega_k x)Y_0(\omega_k x_{k-1})}{J_0(\omega_k x_{k-1})Y_0(\omega_k x_k) - J_0(\omega_k x_k)Y_0(\omega_k x_{k-1})}$
Sphere†	−1/2	$-k_k \dfrac{x_{k-1}}{\sin[\omega_k(x_k - x_{k-1})]}$ $\times \{x\omega_k \cos[\omega_k(x_k - x)] + \sin[\omega_k(x_k - x)]\}$	$k_k \dfrac{x_k}{\sin[\omega_k(x_k - x_{k-1})]}$ $\times \{x\omega_k \cos[\omega_k(x - x_{k-1})] - \sin[\omega_k(x - x_{k-1})]\}$

† $x_0 > 0$; see footnote to Table 1.

and $V_k(\mu, x)$, equations (4c) and (4d), implies that

$$U_k(\mu, x_{k-1}) = 1; \quad U_k(\mu, x_k) = 0, \tag{4e}$$

$$V_k(\mu, x_{k-1}) = 0; \quad V_k(\mu, x_k) = 1. \tag{4f}$$

First consider the case when $h_k \to \infty$, $k = 1, 2, \ldots, (n-1)$. If the solution, equation (4a), should satisfy the boundary conditions, equations (3b), (3d)–(3f), one has, respectively,

$$[\alpha_0/\beta_0 - P_1(\mu, x_0)]\Psi_0^* + P_1(\mu, x_1)\Psi_1^* = 0, \tag{5a}$$

$$P_k(\mu, x_k)\Psi_{k-1}^* + [Q_k(\mu, x_k) - P_{k+1}(\mu, x_k)]\Psi_k^* + P_{k+1}(\mu, x_{k+1})\Psi_{k+1}^* = 0, \quad k = 1, 2, \ldots, (n-1), \tag{5b}$$

$$P_n(\mu, x_n)\Psi_{n-1}^* + [Q_n(\mu, x_n) + (\alpha_n/\beta_n)]\Psi_n^* = 0 \tag{5c}$$

where

$$P_k(\mu, x) = k_k x^{1-2m} U_k'(\mu, x), \tag{5d}$$

$$Q_k(\mu, x) = k_k x^{1-2m} V_k'(\mu, x). \tag{5e}$$

In the derivation of equations (5), the relation

$$P_k(\mu, x_k) + Q_k(\mu, x_{k-1}) = 0 \tag{6}$$

was substantial.

To prove the relation, equation (6), one can write down equation (3a) for the two linearly independent solutions $u_k(\mu, x)$ and $v_k(\mu, x)$

$$[x^{1-2m}u_k'(\mu, x)]' + \omega_k^2 x^{1-2m} u_k(\mu, x) = 0, \tag{7a}$$

$$x^{1-2m}v_k'(\mu, x)]' + \omega_k^2 x^{1-2m} v_k(\mu, x) = 0. \tag{7b}$$

Equation (7a) is then multiplied by $v_k(\mu, x)$, while equation (7b) was multiplied by $u_k(\mu, x)$. The two results are subtracted and the final result is integrated over the interval (x_{k-1}, x_k). Thus, one has

$$x_k^{1-2m}[u_k'(\mu, x_k)v_k(\mu, x_k) - u_k(\mu, x_k)v_k'(\mu, x_k)] - x_{k-1}^{1-2m}[u_k'(\mu, x_{k-1})v_k(\mu, x_{k-1}) - u_k(\mu, x_{k-1})v_k'(\mu, x_{k-1})] = 0. \tag{8}$$

From equations (4c), (4d), (5d), (5e) and (8), one proves the validity of equation (6). Table 3 shows the functions $P_k(\mu, x)$ and $Q_k(\mu, x)$ for the cases of slab, cylindrical and spherical geometries.

The system of equations, equations (5), will form the basis of the analysis for the computation of the eigenvalues and the eigenfunctions of the eigenvalue problem defined by equations (3). Therefore, one needs the values of the functions $P_k(\mu, x)$ and $Q_k(\mu, x)$ at the end points $x = x_{k-1}$ and $x = x_k$ for each subregion k, $k = 1, 2, \ldots, n$. These values can easily be obtained from Table 3. For the case of cylindrical geometry ($m = 0$) the use of the Wronskian relationship leads to a simplification of the coefficients $P_k(\mu, x_k)$ and $Q_k(\mu, x_{k-1})$.

Equations (5) form a linear system of $(n+1)$ homogeneous equations for the determination of Ψ_k^*, $k = 0, 1, 2, \ldots, n$. These equations can be represented in matrix form as

$$[K(\mu)]\{\Psi^*\} = 0 \tag{9a}$$

where

$$\{\Psi^*\}^T = \{\Psi_0^*, \Psi_1^*, \Psi_2^*, \ldots, \Psi_n^*\} \tag{9b}$$

is the transpose of $\{\Psi^*\}$ and

$$[K(\mu)] = \begin{bmatrix} a_0 & b_1 & 0 & 0 & 0 & \cdots & & & & & & & & \\ b_1 & a_1 & b_2 & 0 & 0 & \cdots & & & & & & & & \\ 0 & b_2 & a_2 & b_3 & 0 & \cdots & & & & & & & & \\ \cdots & & & & & & & & & & & & & \\ \cdots & & & & & 0 & b_{k-1} & a_{k-1} & b_k & 0 & \cdots & & & \\ \cdots & & & & & & & & & & & & & \\ \cdots & & & & & & & & & & 0 & b_{n-1} & a_{n-1} & b_n \\ \cdots & & & & & & & & & & 0 & 0 & b_n & a_n \end{bmatrix} \tag{9c}$$

where

$$a_0 = \alpha_0/\beta_0 - P_1(\mu, x_0), \tag{9d}$$

$$b_k = P_k(\mu, x_k); \quad k = 1, 2, \ldots, n, \tag{9e}$$

$$a_k = Q_k(\mu, x_k) - P_{k+1}(\mu, x_k); \quad k = 1, 2, \ldots, (n-1), \tag{9f}$$

$$a_n = Q_n(\mu, x_n) + \alpha_n/\beta_n. \tag{9g}$$

If the system of equations, equation (9a), has a nontrivial solution, the determinant of the matrix $[K(\mu)]$, equations (9c)–(9g), should vanish

$$\det([K(\mu)]) = 0. \tag{9h}$$

The transcendental equation, equation (9h), has an infinite number of real roots, which are the eigenvalues of the eigenvalue problem, defined by equations (3). The standard methods for the solution of equation (9h) always carry the risk of missing some of the eigenvalues in the course of computation; for large values of n the estimation of the determinant of $[K(\mu)]$ can also lead to numerical instabilities. Therefore, in the next section a new approach will be described for the computation of the eigenvalues of the eigenvalue problem, equations (3).

5. PROCEDURE FOR THE COMPUTATION OF THE EIGENVALUES

Wittrick and Williams [1, 2] developed an efficient procedure for the computation of the eigenfrequencies and the buckling loads of linear elastic skeletal structures. Since equation (9a) is presented in the same form as the one utilized by Wittrick and Williams, it is not difficult to adapt their procedure to compute the eigenvalues of the problem considered here.

Wittrick and Williams [1, 2] have shown that the number of positive eigenvalues, $N(\tilde{\mu})$, lying between zero and some prescribed positive value $\mu = \tilde{\mu}$ of the eigenvalue parameter μ, is equal to

$$N(\tilde{\mu}) = N_0(\tilde{\mu}) + s([K(\tilde{\mu})]) \tag{10}$$

where $N_0(\tilde{\mu})$ is the total number of positive eigenvalues not exceeding $\tilde{\mu}$ when all the components of the vector $\{\Psi^*\}$ corresponding to $\tilde{\mu}$ are zero and $s([K(\tilde{\mu})])$ denotes the 'sign-count' of the matrix $[K(\tilde{\mu})]$ as defined in refs. [1, 2].

To find $N_0(\tilde{\mu})$, one takes into account the fact that when all the components of the vector $\{\Psi^*\}$ are zero, the system of equations, equation (3a), degenerates into a decoupled set, namely

$$(x^{1-2m}[\Psi_k'(\mu, x)])' + \omega_k^2 x^{1-2m} \Psi_k(\mu, x) = 0 \tag{11a}$$

in $x_{k-1} < x < x_k$, $k = 1, 2, \ldots, n$, subject to the boundary conditions

$$\Psi_k(\mu, x_{k-1}) = 0, \tag{11b}$$

$$\Psi_k(\mu, x_k) = 0. \tag{11c}$$

The eigencondition of the problem, defined by equations (11), is

$$u_k(\mu, x_{k-1}) v_k(\mu, x_k) - u_k(\mu, x_k) v_k(\mu, x_{k-1}) = 0. \tag{12a}$$

This transcendental equation must first be solved for each layer k, $k = 1, 2, \ldots, n$ to evaluate the number of eigenvalues, $N_{0k}(\tilde{\mu})$, not exceeding $\tilde{\mu}$, for it. Then, the total number of positive eigenvalues, $N_0(\tilde{\mu})$, for the entire multilayered composite can be evaluated as

$$N_0(\tilde{\mu}) = \sum_{k=1}^{n} N_{0k}(\tilde{\mu}). \tag{12b}$$

For the cases of slab ($m = 1/2$) and sphere ($m = -1/2$) an explicit relation for the determination of $N_{0k}(\tilde{\mu})$ is available, namely

$$N_{0k}(\tilde{\mu}) = \text{int}[\omega_k(x_k - x_{k-1})/\pi] \tag{12c}$$

where the symbol 'int (z)' denotes the largest integer not exceeding the value of the argument z of the function.

For the case of a cylinder there is no explicit formula of the kind of equation (12c). However, this difficulty can be alleviated if one considers the fact that for $m = 0$, equation (12a) has the form

$$J_0(\omega_k x_{k-1}) Y_0(\omega_k x_k) - J_0(\omega_k x_k) Y_0(\omega_k x_{k-1}) = 0 \tag{12d}$$

and that the roots of this equation can be computed using standard techniques and then stored in the memory of the computer. Thus, for any specified value of $\tilde{\mu}$, the number of positive eigenvalues $N_{0k}(\tilde{\mu})$, lying in the range $0 < \mu < \tilde{\mu}$, for each layer can be determined and $N_0(\tilde{\mu})$ can be evaluated according to equation (12b).

The 'sign-count' $s([K(\tilde{\mu})])$ is shown [1, 2] to be equal to the number of negative elements along the main diagonal of the matrix $[K^{\triangle}(\tilde{\mu})]$, which is the triangulated form of the matrix $[K(\tilde{\mu})]$, or, equivalently, the 'sign-count' is equal to the number of negative elements in the sequence $d_1/d_0, d_2/d_1, \ldots, d_n/d_{n-1}$, where

$$d_k = d_{k-1}(a_k + a_{k-1}) - d_{k-2} b_{k-1}^2; \quad k = 2, 3, \ldots, n \tag{13}$$

where a_k and b_k are defined by equations (9d)–(9g), and $d_0 = 1$, $d_1 = a_0$.

The case of boundary conditions of the first kind at $x = x_0$ or $x = x_n$, or both, corresponds to $\beta_0 = 0$ or $\beta_n = 0$ or $\beta_0 = \beta_n = 0$, thus resulting in $a_0 \to \infty$ or $a_n \to \infty$ or $a_0 \to \infty$ and $a_n \to \infty$ simultaneously, respectively. In any one of these cases, one simply neglects the corresponding row and column of $[K(\mu)]$, as they do not influence the elimination process described by equation (13) and imply that

$$\Psi_1(\tilde{\mu}, x_0) = 0 \quad \text{or} \quad \Psi_n(\tilde{\mu}, x_n) = 0$$

$$\text{or} \quad \Psi_1(\tilde{\mu}, x_0) = \Psi_n(\tilde{\mu}, x_n) = 0,$$

respectively.

When both terms on the RHS of equation (10) are computed as described in the above, the quantity $N(\tilde{\mu})$ becomes available and the following algorithm can be used to find the eigenvalues μ_i, $i = 1, 2, 3, \ldots$ for the system under consideration:

Step 1. Prescribe the following quantities: I, the total number of eigenvalues μ_i, $i = 1, 2, \ldots, I$ to be computed; I_0, the maximum admissible number of

iterations to compute any eigenvalue μ_i; A, the accuracy of the eigenvalues desired.

Step 2. Prescribe upper bound $\mu_u = 0$, lower bound $\mu_l = 0$ and the initial value of the increment r of the eigenvalue parameter μ; set the eigenvalue counter $i = 1$.

Step 3. Set the iteration counter $i_0 = 1$.

Step 4. Set $r = 2r$ and $\mu_u = \mu_u + r$. Compute $N(\mu_u)$. If $N(\mu_u) < i$, go to *Step* 5, else go to *Step* 6.

Step 5. Set $\mu_l = \mu_u$ and $i_0 = i_0 + 1$. If $i_0 \leqslant I_0$ go to *Step* 4, else go to *Step* 12.

Step 6. If abs $(\mu_u - \mu_l) \leqslant A$ go to *Step* 11, else go to *Step* 7.

Step 7. Set $\tilde{\mu} = (\mu_u + \mu_l)/2$ and compute $N(\tilde{\mu})$. If $N(\tilde{\mu}) < i$, go to *Step* 8, else go to *Step* 9.

Step 8. Set $\mu_l = \tilde{\mu}$ and go to *Step* 10.

Step 9. Set $\mu_u = \tilde{\mu}$ and go to *Step* 10.

Step 10. Set $i_0 = i_0 + 1$. If $i_0 \leqslant I_0$ go to *Step* 6, else go to *Step* 12.

Step 11. Set $\mu_i = (\mu_u + \mu_l)/2$ and store μ_i. Set $\mu_u = \mu_i$, $\mu_l = \mu_i$ and $i = i + 1$. If $i \leqslant I$ go to *Step* 3, else go to *Step* 13.

Step 12. This is an error output indicating that convergence has not been achieved in the sense of the values of I_0 and A prescribed.

Step 13. This is the successful output of the algorithm for the computation of the eigenvalues which is to be followed by the computation of the corresponding eigenfunctions.

From the description of the above algorithm it is obvious that in the course of the iterative process of converging upon some eigenvalue μ_i the choice of $\tilde{\mu}$ is quite random. Thus one is not guaranteed that some value of $\tilde{\mu}$ will not be close enough to some zero of the denominator of the functions $P_k(\mu, x)$ and $Q_k(\mu, x)$, equations (5d) and (5e) or that division by zero will not occur in the course of the elimination process, described by equation (13). In both cases a slight change of the value of $\tilde{\mu}$ enables one to continue the execution of the algorithm discussed. It should also be noted that this multilayer algorithm is not suitable for the computation of the eigenvalues of a single layer problem, subject to boundary conditions of the first kind at both boundaries.

6. CONTACT RESISTANCE AT THE INTERFACE

The foregoing analysis and the computational scheme described are directly applicable if all the layers are in perfect thermal contact. However, the above procedure can readily accomodate the situation that involves contact conductances between any two adjacent layers as now described.

Suppose there is contact conductance at the interface $x = x_k$ or equivalently, between the layers k and $k+1$. Assume that the presence of the interface resistance can be replaced by a fictitious layer of thickness $l^*_{k,k+1}$, which is so small that its capacitance can be neglected. Then the eigenvalue equation, equation (3a), in the local coordinate x^* and for $\alpha^*_{k,k+1} \to \infty$ degenerates into the following equation:

$$\Psi''_{k,k+1}(x^*) = 0 \tag{14a}$$

in $0 < x^* < l^*_{k,k+1}$. The integration of this equation, first from 0 to x^* and then from x^* to $l^*_{k,k+1}$, after transforming to the system coordinates, yields the following expression:

$$(k^*/l^*)_{k,k+1}[\Psi(\mu, x_{k+1}) - \Psi(\mu, x_k)] = k_k \Psi'_k(\mu, x_k). \tag{14b}$$

A comparison of this result with equation (3c) reveals that the two results coincide if

$$h_k = (k^*/l^*)_{k,k+1}. \tag{14c}$$

Equation (14c) implies that the presence of a contact conductance h_k is equivalent to a fictitious layer with $\alpha^* \to \infty$ or $\omega^* \to 0$. The functions $P(\mu, x)$ and $Q(\mu, x)$ for such a layer are determined from the results in Table 3, for $m = 1/2$, by setting $\omega^*_k \to 0$ and $x_x - x_{k-1} = l^*$. One obtains

$$P^*(\mu, x_{k+1}) = P^*(\mu, x_k) = -h_k, \tag{15a}$$

$$Q^*(\mu, x_{k-1}) = Q^*(\mu, x_k) = h_k. \tag{15b}$$

The computational procedure for this case is now straightforward. Whenever there is a contact conductance at some interface $x = x_k$, its presence is replaced by a fictitious layer for which the functions P^* and Q^* are taken as defined by equations (15). In the case $h_k \to \infty$, one has $P^* = Q^* \to \infty$ for the fictitious layer and as a result, by equation (9f), $a_k \to \infty$ for it. In such a case one neglects the corresponding row and column of $[K(\tilde{\mu})]$ in equation (9c) as they do not influence the computational process.

Once the eigenvalues are computed, the determination of the eigenfunction can be carried out making use of equations (5a)–(5c), i.e.

$$\Psi^*_1 = [P_1(\mu, x_0) - \alpha_0/\beta_0]\Psi^*_0/P_1(\mu, x_1), \tag{15c}$$

$$\Psi^*_{k+1} = \{[P_{k+1}(\mu, x_k) - Q_k(\mu, x_k)]\Psi^*_k - P_k(\mu, x_k)\Psi^*_{k-1}\} \div P_{k+1}(\mu, x_{k+1}), \quad k = 1, 2, \ldots, (n-1). \tag{15d}$$

Since the eigenfunctions are arbitrary within a multiplication constant, one can assume that $\Psi^*_0 = \beta_0$ and then compute consequently Ψ^*_k, $k = 1, 2, \ldots, n$ from equations (15c) and (15d). For boundary conditions of the first kind $\beta_0 = 0$ and $\Psi^*_1 = 1$ can be accepted to start the computation of the other components of the vector $\{\Psi^*\}$.

To find the norm of the eigenfunctions one makes use of the relation [8]

$$\int xW_m^2(\lambda x)\,dx = \frac{x^2}{2}\left\{\left(1-\frac{m^2}{\lambda^2x^2}\right)W_m^2(\lambda x) + \tfrac{1}{2}[W_m'(\lambda x)]^2\right\} \quad (16)$$

where W_m is any linear combination of Bessel functions.

The integrals defined by equation (2b) are rewritten in a more convenient form as

$$\int_{x_{k-1}}^{x_k} x^{1-2m}\Psi_{ik}^2(\mu_i,x)\,dx = \int_{x_{k-1}}^{x_k} x\left[\frac{\Psi_{ik}(\mu_i,x)}{x^m}\right]^2 dx. \quad (17)$$

Comparison between the RHS of equation (17) and the LHS of equation (16) yields

$$W_{mi}(\omega_k x) = x^{-m}\Psi_{ik}(\mu_i,x). \quad (18)$$

The differentiation of equation (18) leads one to

$$W_{mi}'(\omega_k x) = x^{-2m}[x^m\Psi_{ik}'(\mu_i,x) - mx^{m-1}\Psi_{ik}(\mu_i,x)]. \quad (19a)$$

The last relation can be rearranged as

$$W_{mi}'(\omega_k x) = k_k^{-1}x^{m-1}[\Psi_{k-1}^*P_k(\mu_i,x) + \Psi_k^*Q_k(\mu_i,x)] - mx^{-(m+1)}\Psi_{ik}(\mu_i,x). \quad (19b)$$

Introducing equation (18) and equation (19b) into equation (16), one finally obtains

$$\begin{aligned}\int_{x_{k-1}}^{x_k} x^{1-2m}\Psi_{ik}^2(\mu_i,x)\,dx &= (2\omega_{ik}^2)^{-1}[x_k^{-2m}(x_k^2\omega_{ik}^2-m^2)\Psi_k^{*2} - x_{k-1}^{-2m}(x_{k-1}^2\omega_{ik}^2-m^2)\Psi_{k-1}^{*2}] \\ &+(2\omega_{ik}^2)^{-1}x_k^{-2m}\{k_k^{-1}x_k^{2m}[\Psi_{k-1}^*P_k(\mu_i,x_k) + \Psi_k^*Q_k(\mu_i,x_k)] - m\Psi_k^*\}^2 \\ &-(2\omega_{ik}^2)^{-1}x_{k-1}^{-2m}\{k_k^{-1}x_{k-1}^{2m}[\Psi_{k-1}^*P_k(\mu_i,x_{k-1}) + \Psi_k^*Q_k(\mu_i,x_{k-1})] - m\Psi_{k-1}^*\}^2. \end{aligned} \quad (20)$$

Equation (20) allows one to estimate the normalization integral defined by equation (2b) and then estimate the normalized eigenfunction

$$\tilde{\Psi}_{ik}(\mu_i,x) = \Psi_{ik}(\mu_i,x)N_i^{-1/2}. \quad (21)$$

Finally, equations (2d) and (2e) take the form

$$\Omega_i(x_0) = \{\Psi_0^*[1+P_1(\mu_i,x_0)] - \Psi_1^*P_1(\mu_i,x_1)\}(\alpha_0+\beta_0)^{-1}, \quad (22a)$$

$$\Omega_i(x_n) = \{\Psi_n^*[1-Q_n(\mu_i,x_n)] - \Psi_{n-1}^*P_n(\mu_i,x_n)\}(\alpha_n+\beta_n)^{-1}. \quad (22b)$$

7. THE SOLUTION OF THE ORIGINAL PROBLEM

The solution of the initial-boundary value problem, defined by equations (1), was written down in the general form, equation (2a). Now it will be written explicitly for the cases of multilayered slabs, cylinders and spheres. In a form adequate for direct computations, one has

$$\begin{aligned}T_k(x,t) = &\left\{\left[\beta_n+\alpha_n\left(\frac{x_k-x}{k_k}+\sum_{i=k+1}^{n}\frac{x_i-x_{i-1}}{k_i}+\sum_{i=k}^{n-1}\frac{1}{h_i}\right)\right]\Phi_0 \right. \\ &\left.+\left[\beta_0+\alpha_0\left(\frac{x-x_{k-1}}{k_k}+\sum_{i=1}^{k-1}\frac{x_i-x_{i-1}}{k_i}+\sum_{i=1}^{k-1}\frac{1}{h_i}\right)\right]\Phi_n\right\} \\ &\times\left[\alpha_0\beta_n+\alpha_n\beta_0+\alpha_0\alpha_n\left(\sum_{k=1}^{n}\frac{x_k-x_{k-1}}{k_k}+\sum_{k=1}^{n-1}\frac{1}{h_k}\right)\right]^{-1} \\ &+\sum_{i=1}^{\infty}\frac{\Psi_{ik}(\mu_i,x)}{N_i}\exp(-\mu_i^2t)\{f_i-\mu_i^{-2}[\Phi_0\Omega_i(x_0)+\Phi_n\Omega_i(x_n)]\} \end{aligned} \quad (23a)$$

for $m = 1/2$,

$$\begin{aligned}T_k(x,t) = &\left\{\left[\beta_n+\alpha_n\left(\frac{1}{k_k}\ln\frac{x_k}{x}+\sum_{i=k+1}^{n}\frac{1}{k_i}\ln\frac{x_i}{x_{i-1}}+\sum_{i=k}^{n-1}\frac{1}{h_ix_i}\right)\right]\Phi_0 \right. \\ &\left.+\left[\beta_0+\alpha_0\left(\frac{1}{k_k}\ln\frac{x}{x_{k-1}}+\sum_{i=1}^{k-1}\frac{1}{k_i}\ln\frac{x_i}{x_{i-1}}+\sum_{i=1}^{k-1}\frac{1}{h_ix_i}\right)\right]\Phi_n\right\} \\ &\times\left[\alpha_0\beta_n+\alpha_n\beta_0+\alpha_0\alpha_n\left(\sum_{k=1}^{n}\frac{1}{k_k}\ln\frac{x_k}{x_{k-1}}+\sum_{k=1}^{n-1}\frac{1}{h_kx_k}\right)\right]^{-1} \\ &+\sum_{i=1}^{\infty}\frac{\Psi_{ik}(\mu_i,x)}{N_i}\exp(-\mu_i^2t)\{f_i-\mu_i^{-2}[\Phi_0\Omega_i(x_0)+\Phi_n\Omega_i(x_n)]\} \end{aligned} \quad (23b)$$

for $m = 0$, and

$$T_k(x,t) = \left[\left\{\beta_n + \alpha_n\left[\frac{1}{k_k}\left(\frac{1}{x}-\frac{1}{x_k}\right)+\sum_{i=k+1}^{n}\frac{1}{k_i}\left(\frac{1}{x_{i-1}}-\frac{1}{x_i}\right)+\sum_{i=k}^{n-1}\frac{1}{h_i x_i^2}\right]\right\}\Phi_0\right.$$
$$\left.+\left\{\beta_0+\alpha_0\left[\frac{1}{k_k}\left(\frac{1}{x_{k-1}}-\frac{1}{x}\right)+\sum_{i=1}^{k-1}\frac{1}{k_i}\left(\frac{1}{x_{i-1}}-\frac{1}{x_i}\right)+\sum_{i=1}^{k-1}\frac{1}{h_i x_i^2}\right]\right\}\Phi_n\right]$$
$$\times\left\{\alpha_0\beta_n+\alpha_n\beta_0+\alpha_0\alpha_n\left[\sum_{k=1}^{n}\frac{1}{k_k}\left(\frac{1}{x_{k-1}}-\frac{1}{x_k}\right)+\sum_{k=1}^{n-1}\frac{1}{h_k x_k^2}\right]\right\}^{-1}$$
$$+\sum_{i=1}^{\infty}\frac{\Psi_{ik}(\mu_i,x)}{N_i}\exp(-\mu_i^2 t)\{f_i-\mu_i^{-2}[\Phi_0\Omega_i(x_0)+\Phi_n\Omega_i(x_n)]\} \tag{23c}$$

for $m = -1/2$.

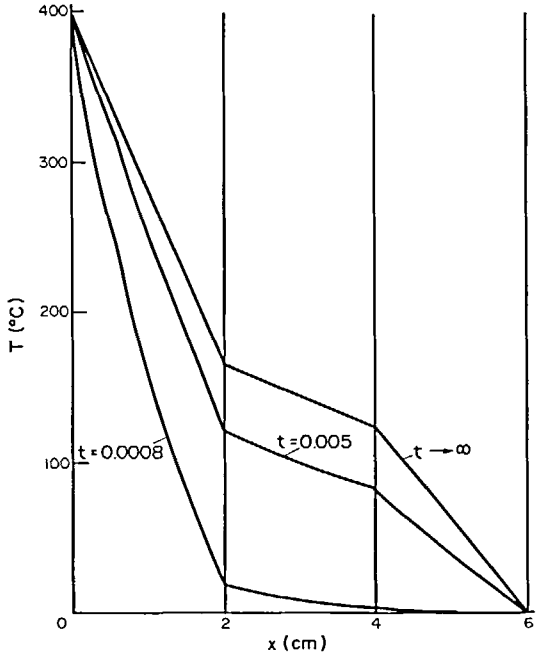

FIG. 2. Temperature distribution for various times (the example of ref. [10]).

Once the eigenvalues and the eigenfunctions, μ_i and $\Psi_{ik}(\mu_i, x)$, respectively, have been computed as was described in the previous sections, no difficulty is encountered in evaluating the solutions, equations (23), having in mind also Tables 1–3.

8. ILLUSTRATIVE EXAMPLES AND CONCLUDING REMARKS

To illustrate the applicability of the solutions derived and the method proposed, a number of problems were solved. The code prepared for this purpose was first tested on several single layered problems with solutions that were previously known. Then the same problems were modelled as multilayer problems, the adjacent layers having identical physical properties. A satisfactory coincidence (approximately of the order of the accuracy of the eigenvalues computed) was observed in the computation of the corresponding eigenfunctions and temperature distributions.

Figure 2 shows some temperature distributions for the numerical example discussed in detail in ref. [10].

Figures 3 and 4 show similar distributions for the cases of cylindrical ($m = 0$) and spherical ($m = -1/2$)

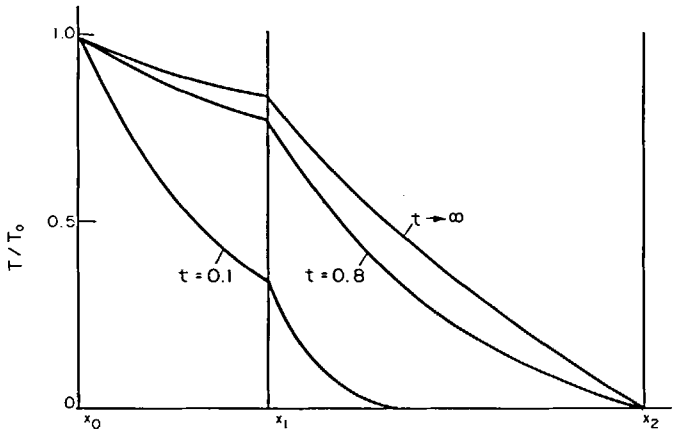

FIG. 3. Temperature distributions for a two layered cylinder for various times: $\alpha_1^* = 5, x_0 = 1, x_1 = 2, x_2 = 4$, $\alpha_2^* = 1, m = 0, \alpha_0 = \alpha_2 = 1, \beta_0 = \beta_2 = 0, \Phi_0 = T_0, \Phi_2 = 0, T_k(x,0) = 0, k = 1,2$.

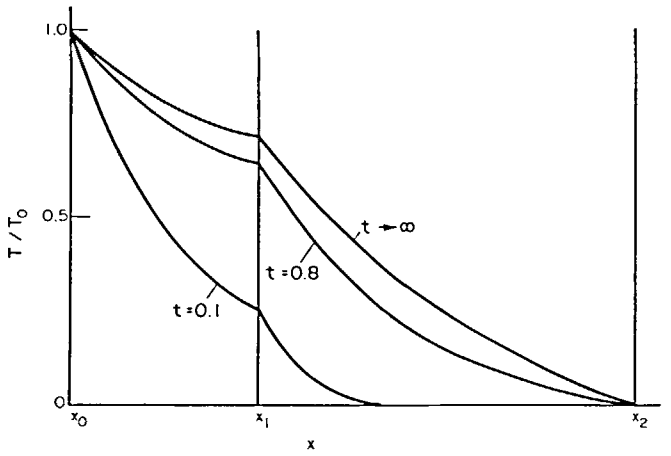

FIG. 4. Temperature distributions for a two layered sphere for various times: $\alpha_1^* = 5, \alpha_2^* = 1, x_0 = 1, x_1 = 2, x_2 = 4, m = -1/2, \alpha_0 = \alpha_2 = 1, \beta_0 = \beta_2 = 0, \Phi = T_0, \Phi_2 = 0, T_k(x,0) = 0, k = 1,2.$

geometries, respectively the verification of the results on Figs. 2–4 was obtained following the testing strategy mentioned above.

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DIFFUSION DANS DES COUCHES COMPOSITES ET SOLUTION AUTOMATIQUE DU PROBLEME DES VALEURS PROPRES

Résumé—Le traitement analytique du problème de la conduction thermique unidirectionnelle dans des composites multicouches nécessite la solution d'un problème de valeurs propres si cette solution analytique doit être appliquée à des buts pratiques. Un tel problème de valeur propre n'est pas du type conventionnel Sturm–Liouville à cause des discontinuités des fonctions. Sa solution avec les techniques conventionnelles n'est pas garantie de l'absence de valeurs propres au cours du calcul.

Une solution analytique d'un problème variable de conduction dans des plaques, des cylindres et des sphères multicouches est présentée qui applique un algorithme correct pour le calcul automatique des valeurs propres et des fonctions propres du système résultant de type Sturm–Liouville.

DIFFUSION IN GESCHICHTETEN KÖRPERN MIT AUTOMATISCHER LÖSUNG DES EIGENWERTPROBLEMS

Zusammenfassung—Die analytische Behandlung instationärer Probleme der eindimensionalen Wärmeleitung in mehrschichtigen Körpern nach dem Verfahren der orthogonalen Reihenentwicklung erfordert die Lösung eines entsprechenden Eigenwertproblems, wenn die analytische Lösung auf praktische Probleme angewendet werden soll. Solche Eigenwertprobleme sind nicht vom normalen Sturm–Liouville-Typ, da die Koffizienten-Funktionen Diskontinuitäten enthalten. Die Lösung nach konventionellen Verfahren ist wegen fehlender Eigenwerte im Verlauf der Berechnung nicht immer möglich. Eine analytische Lösung instationärer Probleme der eindimensionalen Wärmeleitung in mehrschichtigen Platten, Zylindern und Kugeln wird angegeben, die einen sicheren Algorithmus für die automatische Berechnung der Eigenwerte und der Eigenfunktionen des resultierenden Systems vom Sturm–Liouville-Typ enthält.

ИССЛЕДОВАНИЕ ДИФФУЗИИ В КОМПОЗИТНЫХ СЛОЯХ НА ОСНОВАНИИ РЕШЕНИЯ ЗАДАЧИ НА СОБСТВЕННЫЕ ЗНАЧЕНИЯ

Аннотация—Аналитическое исследование задач нестационарной теплопроводности в одномерных многослойных композитных средах методом ортогонального разложения требует решения соответствующей задачи на собственные значения в случае, если оно необходимо для практических целей. Такая задача не является обычной задачей типа Штурма-Лиувилля из-за наличия разрывов в коэффициентных функциях. Ее решение обычными методами не гарантируется ввиду отсутствия собственных значений в процессе расчета. Представлено аналитическое решение одной задачи нестационарной теплопроводности в одномерных многослойных плитах, цилиндрах и сферах, которое дает надежный алгоритм для автоматического расчета собственных значений и функций системы уравнений типа Штурма-Лиувилля.